Middle East Technical University Department of Computer Engineering

CENG 567

Design and Analysis of Algorithms

Fall 2020-2021

Homework 1 - Stable Matching & Algorithm Analysis

Due Date:

Instructions. You may work with other students, but you must individually write your solutions **in your own words**. If you work with other students or consult outside sources(such as Internet/book), cite your sources.

Submissions. Submit a pdf file through odtuclass. LaTeX or Word typed submission is required.

1. **Stable Matching**
(a) Use Gale-Shapley algorithm to find a stable matching for the following set of four colleges, four students and their preference lists.

College	Preference List
C1	S2, S1, S4, S3
C2	S2, S1, S3, S4
C3	S1, S2, S3, S4
C4	S1, S3, S2, S4

Student	Preference List
S1	C4, C2, C3, C1
S2	C1, C4, C2, C3
S3	C1, C2, C3, C4
S4	C4, C3, C1, C2

(b) Find another stable matching with the same algorithm.

(c) Consider a pair of man m and woman w where m has w at the top of his preference list and w has m at the top of her preference list. Does it always have to be the case that the pairing (m, w) exist in every possible stable matching? If it is true, give a short explanation. Otherwise, give a counterexample.

(d) Give an instance of n colleges, n students, and their preference lists so that the Gale-Shapley algorithm requires only $O(n)$ iterations, and prove this fact.

(e) Give another instance for which the algorithm requires $\Omega(n^2)$ iterations (that is, it requires at least cn^2 iterations for some constant $0 < c \leq 1$), and prove this fact.

2. **Stable Matching Variation**
Consider a Stable Matching problem with men and women. Consider a woman w where she prefers man m to m', but both m and m' are low on her list of preferences. Can it be the case that by switching the order of m and m' on her list of preferences (i.e., by falsely claiming that she prefers m' to m) and

running the algorithm with this modified preference list, w will end up with a man m'' that she prefers to both m and m'? Either give a proof that shows such an improvement is impossible, or give an example preference lists for which an improvement for w is possible.

3. **Asymptotics**
What is the running time of this algorithm as a function of n? Specify a function f such that the running time of the algorithm is $\Theta(f(n))$. (Give a detailed answer.)

Algorithm 1:

```
i = 2;
while i < n do
    j = 1;
    while j < n do
        Some Θ(1) operation ;
        j = j * i ;
    end
    i = i + 1 ;
end
```

Note that you can ignore loop counter operations.

4. **Big O and Ω**

a) Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove the following conjectures.

- $f(n) = O(g(n))$ implies $g(n) = O(f(n))$
- $f(n) = O((f(n))^2)$
- $f(n) + o(f(n)) = \Theta(f(n))$

b) For each function $f(n)$ below, find (and prove that) (1) the smallest integer constant H such that $f(n) = O(n^H)$, and (2) the largest positive real constant L such that $f(n) = \Omega(n^L)$. Otherwise, indicate that H or L do not exist. Note that all logarithms are base 2.

- $f(n) = \frac{n(n+1)}{2}$
- $f(n) = \sum_{k=0}^{\lceil logn \rceil} \frac{n}{2^k}$
- $f(n) = n(logn)^2$